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# Propagation of long waves over water of slowly varying depth

J. HARBAND

*University of the Negev, Beersheva, Israel*

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SUMMARY
Asymptotic expressions are obtained for the refraction, reflection and modulation of long waves progressing over water of variable depth when the rate of depth variation is small compared with the wavelength. The two- and three-dimensional cases are treated.

## Introduction

When long waves progress over water of variable depth they are transformed under the influence of the bottom features: the wave crests are retarded over regions of shallower water and they advance over deeper water. As a result, the waves change direction and are bent in such a manner that they tend to assume the shape of the depth contours [1]. This effect is called wave refraction by oceanographers. If the bottom is two-dimensional, as in the case of long waves propagating down a canal, an analogous phenomenon occurs: the waves, in general, decompose into reflected and transmitted components each of which undergoes a modulation in amplitude, wave number and phase. An important problem for applications, as well as an interesting one mathematically, is to obtain explicit expressions for these phenomena when the depth varies in a general way. This problem, when formulated in terms of the exact linear theory of water waves, has proved difficult to solve, and, hence, researchers have relied on various approximate methods.

The purpose of the present article is to present a method of solution which leads to explicit asymptotic expressions for the refraction, reflection, and modulation of long waves when the rate of variation of the bottom elevation is small compared with the wavelength. In obtaining these results, the bottom surface is assumed to satisfy certain integrability and differentiability conditions but is otherwise unrestricted in shape.

For the two-dimensional problem we find that the reflection coefficient diminishes as the smoothness of the bottom profile increases. The solution of the two-dimensional problem remains valid when the rate of variation of the bottom elevation is of the order of magnitude of the wavelength, and, in fact, our reflection coefficient includes a result of Kreisel [2] for a non-slowly-varying bottom profile. In addition, we find that the phase shift, amplitude, and wave number depend not only on the bottom elevation but on all of the derivatives of the bottom elevation as well. As a result, our expression for the phase shift of the transmitted wave, in the case of no reflection, includes and extends the result obtained from the ray theory (described below) when the latter is applied to shallow water.

For the three-dimensional problem we obtain an asymptotic expression, valid for small variation rate of the bottom elevation, which explicitly exhibits the dependence of the wave phase on the bottom elevation assuming that the latter is twice continuously differentiable in the direction perpendicular to that of the incoming wave, that it has an integrable derivative in the propagation direction, and that it is flat outside a compact region. The method will yield the dependence of the wave phase on the derivatives of the bottom elevation if the bottom is assumed to be smoother than this.

The problem of determining the reflection and modulation of waves over a general two-dimensional bottom profile has been approached in several different ways. The most difficult of these is to attempt a true solution of the exact linear equations of water waves. Kreisel [2], for example, has used conformal mapping to reduce the problem to a linear integral equation and to thus obtain estimates on the reflection coefficient. The estimates grow more precise in the limit as the wavelength grows large with respect to the depth, i.e., for shallow water. A second approach utilizes the approximate "shallow-water" equations of water waves. Here also the main results have concerned reflection. Using a theory of Bremmer [3], Kajiura [4] has determined an approximate reflection coefficient due to a shelf when the transition between the two levels of the shelf is gradual. A third method is the "ray" method (Keller [5], Sverdrup and Munk [6]), in which one obtains a formal asymptotic solution to the exact linear equation which is assumed to be asymptotic to the true solution when the variation rate of the bottom elevation is small compared with the depth and with the wave-length. This method leads to the determination of the local amplitude, wave number, and phase of a progressing wave. It can also be used to determine the reflected wave, but not in a straightforward manner. It has therefore found special application when reflection is negligible, e.g., in the study of waves approaching a beach. The ray method has been extended to the non-linear equations of water-waves by Shen and Keller [7] (see also Chu and Mei [8]). When the bottom is three-dimensional, the only method that seems to have been employed is the ray method.

The distinguishing feature of the present approach is the way in which the magnitude of the bottom elevation, $D$, is defined in terms of the depth and the variation rate of the bottom elevation [cf. 7a, b)]. This assumption enables one to obtain an approximate problem which is linearized about a flat bottom and thus admits an explicit solution.

The present treatment also differs as regards the introduction of the shallow water, or long wave, approximation. In the linear shallow water theory of water waves (Stoker [9]) the shallow water approximation is already embodied in the basic equations of motion; here, we solve an approximate form of the exact linear equation, valid for small $D$, in which the potential equation is retained and only in the steady-state solution is the shallow-water approximation made [cf. 19a, b)].

The steady-state surface shape is most conveniently obtained by following an approach to steady-state water-wave problems used by, e.g., Stoker [9]. According to this approach, one formulates an appropriate initial-value problem whose solution yields, when the time is allowed to go to infinity, the desired steady-state solution. This obviates the need to prescribe radiation conditions on an a-priori unknown steady-state solution. To apply this method to the present problem, we take a bottom which is initially flat but quickly assumes its final varying shape. This enables one to use, as an initial wave, the simple, well-known progressing-wave solution over a flat bottom [cf. 3a, b)]. For large time the transient effect due to the "creation" of the variable bottom decays and one is left with the steady-state solution. It should be mentioned that, in other contexts, such initial-value formulations have been used to generate transient wave motions which are themselves the primary objects of study, e.g., tsunamis (Carrier [10]). After taking the shallow-water limit in the steady-state solution, one is left with an expression for the surface shape which is independent of depth but which depends on the variation rate of the bottom elevation, $\beta$. This expression is evaluated asymptotically for small $\beta$. In two dimensions this evaluation is accomplished simply by integrating by parts. In three dimensions, however, the method of stationary phase for multiple integrals [11] must be employed.

## PART A. FORMULATION OF THE PROBLEM AND DESCRIPTION OF THE APPROXIMATION SCHEME

### A-I. Formulation of the problem

We shall consider a particular infinitesimal, irrotational, three-dimensional wave motion, which we will presently specify, existing in an inviscid, incompressible liquid bounded above by the free surface $y = \eta(X, Z, t)$ and below by the surface $y(X, Z, t) = -h + D\gamma(mX, mZ)a(t)$ where $X$ and $Z$ are horizontal coordinates and $y$ the vertical coordinate. We assume that $\gamma(x, z)$ and $\gamma_x(x, z)$ admit a Fourier transformation with respect to $x$ and $z$. The function $a(t)$ is assumed to be continuously differentiable and is defined by the formula:

$$a(t) = \begin{cases} 0, & t \leqq 0, \\ 1, & t \geqq \alpha > 0, \end{cases} \qquad a'(t) \geqq 0. \tag{1}$$

The function $a(t)$ serves to transform, in the time interval $[0, \alpha]$, the originally flat bottom $y = -h$ into a rigid one whose elevation varies with $X$ and $Z$. The constant $m$ characterizes the spatial variation rate of the bottom elevation and the constant $D$ gives the magnitude of the bottom elevation for $t \geqq \alpha$. Their orders of magnitude are precisely defined in Section A-II below.

Under the above assumptions, any motion is described by a velocity potential function $\Phi(X, y, Z, t)$ satisfying the exact linear system [9]:

$$\Phi_{XX} + \Phi_{yy} + \Phi_{ZZ} = 0, \quad -h + D\gamma a \leqq y \leqq 0, \quad -\infty < t < \infty; \tag{2a}$$

$$\Phi_{tt} + g\Phi_y = 0, \quad y = 0, \quad -\infty < t < \infty; \tag{2b}$$

$$\Phi_y - mDa\gamma_x\Phi_X - mDa\gamma_z\Phi_Z - Da'\gamma = 0, \quad y = -h + D\gamma a, \quad -\infty < t < \infty. \tag{2c}$$

To uniquely specify the motion that we shall consider, we assume that for $t \leqq 0$ it is given by the following particular solution of (2a–c):

$$\Phi(X, y, Z, t) = (Cg/\sigma)\cosh[M(y + h)]\exp[i(MX - \sigma t)], \quad t \leqq 0, \tag{3a}$$

where

$$\sigma^2 = gM\tanh Mh. \tag{3b}$$

The potential (3a, b) represents a wave progressing in the positive $X$ direction over the flat bottom $y = -h$ with the surface shape

$$\eta(X, Z, t) = iC\cosh(Mh)\exp[i(MX - \sigma t)] \tag{4a}$$

or, taking the real part,

$$\eta(X, Z, t) = -C\cosh(Mh)\sin(MX - \sigma t). \tag{4b}$$

We are primarily interested in investigating the steady state progressing wave motion that is established after the transient motion due to the creation of the varying rigid bottom has decayed. For this purpose we formulate an initial value problem for $t \geqq 0$ by adjoining to (2a–c) the following initial conditions obtained from (3a)

$$\Phi(X, y, Z, 0) = (Cg/\sigma)\cosh[M(y+h)]\exp(iMX), \quad -h \leqq y \leqq 0; \tag{5a}$$

$$\Phi_t(X, y, Z, 0) = -iCg\cosh[M(y+h)]\exp(iMX), \quad -h \leqq y \leqq 0. \tag{5b}$$

**A-II. Description of the approximation scheme – Approximation for small bottom elevation**

Our goal is to obtain an explicit solution of the initial value problem (2a–c), (5a, b). The two difficulties preventing this are, first, that the unknown functions $\Phi_X$ and $\Phi_Z$ occur in (2c) multiplied by the functions $\gamma_x$ and $\gamma_z$ and, secondly, that (2c) is not linearized about $y = -h$. We overcome these by replacing (2a–c) by an approximate system obtained by suitably defining the orders of magnitude of the initial depth, $h$, the initial wave amplitude, the rate of variation of the bottom elevation, and the bottom elevation, $D$. As the reference quantity we use the initial wave number, $M$, which we take to be of order $O(1)$.

To characterize these quantities we introduce the following parameters which are assumed to lie in the interval (0, 1):

| *quantity* | *parameter* | *definition* | |
|---|---|---|---|
| initial depth | $\delta$ | $\delta = (Mh)^{\frac{1}{2}}$ | (6a) |
| initial wave amplitude | $\varepsilon$ | $\varepsilon = CM$ | (6b) |
| ratio of initial wave ampl. to initial depth | $\rho$ | $\rho = C/h = \varepsilon/\delta^2$ | (6c) |
| spatial variation rate of bottom elevation | $\beta$ | $\beta = m/M,\ \gamma_x(x, z),\ \gamma_z(x, z) = O(1)$ | (6d) |

In addition, we assume that the magnitude, $D$, of the bottom elevation is related to $\beta$ and $\delta$ in the following way:

$$D = \beta\delta^2/M; \quad (7a) \qquad D = \beta^2\delta^2/M; \ \gamma(x, z) = O(1) \quad (7b)$$

where (7a) will be used in the two dimensional problem of Part B in which $\gamma$ is independent of $Z$, i.e. $\gamma = \gamma(mX)$, and (7b) will be used in the three dimensional problem of Part C.

Assumption (7a), (7b) implies that the ratio of the area (volume) under the curve

$y = D\gamma(mX)$, $-\infty < X < \infty$, (surface $y = D\gamma(mX, mZ)$, $-\infty < X, Z < \infty$) to the initial depth is independent of $\beta$ and $\delta$. This follows from the easily verified formulas:

$$h^{-1}\int_{-\infty}^{\infty} D\gamma(mX)\,dX = (1/M)\int_{-\infty}^{\infty}\gamma(x)\,dx,$$

$$h^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} D\gamma(mX, mZ)\,dX\,dZ = (1/M^2)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\gamma(x,z)\,dx\,dz.$$

We now describe briefly the approximation scheme and the series of asymptotic limiting processes that will be employed in the present treatment. We begin by assuming that the solution, $\Phi$, of (2a–c), (5a, b) can be expressed as a power series in the parameter $D$. Substituting this series into (2a–c), (5a, b), one arrives at an initial value problem which is explicitly solvable. Carrying out the solution, we obtain an expression for the surface shape. We then take the limit as $t \to \infty$; the transient motion dies out and one is left with the steady state surface shape. In the latter expression we take the limit as $\delta \to 0$; more precisely, we take $\delta$ to be much smaller than $\beta$. The resulting expression turns out to be independent of $\delta$ and we study its asymptotic properties with respect to $\beta$.

We remark that when $\delta$ is small $\rho$ must also be taken small. This is necessary in order that the linearized surface condition (2b) remain a good approximation to the original non-linear surface conditions:

$$g\eta + \Phi_t + \tfrac{1}{2}(\Phi_X^2 + \Phi_y^2 + \Phi_Z^2) = 0, \qquad y = \eta, \tag{8a}$$

$$\Phi_X\eta_X + \Phi_Z\eta_Z + \eta_t = \Phi_y, \qquad y = \eta. \tag{8b}$$

One can show this, following Peregrine [12], by substituting (3a), (4a) into (8a, b). In (8a), for example, the first two terms will be of order $\varepsilon$ and, since $\sigma$ is of order $\delta$ [cf. (3b)] the remaining, non-linear, terms will be of order $\varepsilon^2/\delta^2$. Thus, to neglect the latter with respect to the former, one needs $\rho \ll 1$. Accordingly, in what follows, we take $\rho$ to be a sufficiently small but fixed parameter.

The parameter $D$ can be thought of as characterizing the magnitude of the disturbance acting on the initial progressing wave for $t \geqq \alpha$. Accordingly, it is natural to express the solution of (2a–c), (5a, b) as a formal power series in $D$ of the form:

$$\Phi(X, y, Z, t) = (Cg/\sigma)\cosh[M(y+h)]\exp[i(MX - \sigma t)] + \sum_1^{\infty} D^n\psi^{(n)}(X, y, Z, t). \tag{9}$$

If one substitutes (9) into (5a, b), (2a–c), one obtains:

$$\sum_1^{\infty} D^n\psi^{(n)}(X, y, Z, 0) = 0, \qquad -h + D\gamma a < y < 0; \tag{10a}$$

$$\sum_1^{\infty} D^n\psi_t^{(n)}(X, y, Z, 0) = 0, \qquad -h + D\gamma a < y < 0; \tag{10b}$$

$$\sum_1^{\infty} D^n\psi_{XX}^{(n)} + \sum_1^{\infty} D^n\psi_{yy}^{(n)} + \sum_1^{\infty} D^n\psi_{ZZ}^{(n)} = 0, \qquad -h + D\gamma a \leqq y \leqq 0; \tag{10c}$$

$$\sum_{1}^{\infty} D^n \psi_{tt}^{(n)} + g \sum_{1}^{\infty} D^n \psi_y^{(n)} = 0, \quad y = 0; \tag{10d}$$

$$(CgM/\sigma) \sinh(MD\gamma a) \exp[i(MX - \sigma t)] + \sum_{1}^{\infty} D^n \psi_y^{(n)}(X, -h + D\gamma a, Z, t)$$

$$- mD\gamma_x a\{(iCgM/\sigma) \cosh(MD\gamma a) \exp[i(MX - \sigma t)] + \sum_{1}^{\infty} D^n \psi_X^{(n)}(X, -h + D\gamma a, Z, t)\}$$

$$- mD\gamma_z a \sum_{1}^{\infty} D^n \psi_Z^{(n)}(X, -h + D\gamma a, Z, t) - Da'\gamma = 0. \tag{10e}$$

In (10e) we expand each term in powers of $D$ as well as expanding $\psi_X^{(n)}$, $\psi_y^{(n)}$, and $\psi_Z^{(n)}$ with respect to the second variable about $y = -h$. If in the resulting expression one equates the coefficients of like powers of $D$, one obtains a sequence of initial value problems for the $\psi^{(n)}$ each of which is linearized about $y = -h$. We shall treat only the first order problem. Setting $\psi^{(1)} \equiv \psi$ and

$$G = \gamma - i\beta\gamma_x, \tag{11}$$

the first order problem becomes:

$$\psi(X, y, Z, 0) = \psi_t(X, y, Z, 0) = 0, \quad -h \leqq y \leqq 0; \tag{12a}$$

$$\psi_{XX} + \psi_{yy} + \psi_{ZZ} = 0, \quad -h \leqq y \leqq 0; \tag{12b}$$

$$\psi_{tt} + g\psi_y = 0, \quad y = 0; \tag{12c}$$

$$\psi_y = \gamma a' - (\varepsilon gMGa/\sigma) \exp[i(MX - \sigma t)], \quad y = -h. \tag{12d}$$

## PART B. THE TWO-DIMENSIONAL PROBLEM

### B-I. Solution of the initial-value problem

In this part we assume that $\gamma$ is independent of $z$: $\gamma_z = 0$ for all $x$, $z$. Physically, this means that the level lines of the initial surface wave lie over level lines of the bottom elevation. Accordingly, we assume that the solution $\psi$ of (12a–d) has no $z$-dependence; its derivatives with respect to $z$ are assumed to vanish.

In order to obtain a solution of (12a–d) we utilize the Fourier transformation with respect to $X$:

$$\bar{\psi}(\xi, y, t) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \psi(X, y, t) e^{-i\xi X} dX.$$

The initial value problem (12a–d) then becomes

$$\bar{\psi}(\xi, y, 0) = \bar{\psi}_t(\xi, y, 0) = 0, \quad -h \leqq y \leqq 0; \tag{13a}$$

$$-\xi^2 \bar{\psi} + \bar{\psi}_{yy} = 0, \quad -h \leqq y \leqq 0; \tag{13b}$$

$$\bar{\psi}_{tt} + g\bar{\psi}_y = 0, \quad y = 0; \tag{13c}$$

$$\bar{\psi}_y = \bar{\gamma}a' - (\varepsilon gMa/\sigma)\exp(-i\sigma t)\overline{G(mX)\exp(iMX)}, \quad y = -h. \tag{13d}$$

From (13b),

$$\bar{\psi}(\xi, y, t) = A(\xi, t)\cosh\xi y + B(\xi, t)\sinh\xi y, \quad -h \leqq y \leqq 0. \tag{14}$$

Substituting (14) into (13c, d) and eliminating $B$, one gets an ordinary differential equation for $A$ in the variable $t$:

$$A_{tt} + (g\xi\tanh\xi h)A$$
$$= [-g\bar{\gamma}a' + (\varepsilon g^2 Ma/\sigma)\exp(-i\sigma t)\overline{G(mX)\exp(iMX)}]/\cosh\xi h, \quad t > 0. \tag{15a}$$

From (13a) and (14) one deduces the initial conditions

$$A(\xi, 0) = A_t(\xi, 0) = 0. \tag{15b}$$

Denoting $k(\xi) = (g\xi\tanh\xi h)^{\frac{1}{2}}$, the initial value problem (15a, b) has the solution

$$A(\xi, t) = \int_0^t \frac{k^{-1}\sin[k(t-t')]}{\cosh\xi h}$$
$$\times \{-g\bar{\gamma}a'(t') + (\varepsilon g^2 Ma/\sigma)\exp(-i\sigma t')\overline{G(mX)\exp(iMX)}\}\,dt'.$$

We integrate by parts with the first term in brackets using (1) to obtain

$$A(\xi, t) = -\frac{g\bar{\gamma}}{\cosh\xi h}\left\{k^{-1}\sin[k(t-\alpha)] + \int_0^\alpha \cos[k(t-t')]a(t')\,dt'\right\}$$
$$+ \frac{(\varepsilon g^2 M/\sigma)\overline{G(mX)\exp(iMX)}}{\cosh\xi h}$$
$$\times \int_0^t k^{-1}\sin[k(t-t')]\exp(-i\sigma t')a(t')\,dt'.$$

We now take $\alpha$ in (1) to be of the order of magnitude of the quantities that we have heretofore neglected and write

$$A(\xi, t) = -\frac{g\bar{\gamma}k^{-1}\sin kt}{\cosh\xi h}$$
$$+ \frac{(\varepsilon g^2 M/\sigma)\overline{G(mX)\exp(iMX)}}{\cosh\xi h}\int_0^t k^{-1}\sin[k(t-t')]\exp(-i\sigma t')\,dt'. \tag{16}$$

To evaluate the transform of a product which appears in (16) we make use of the convolution theorem in the form

$$(2\pi)^{\frac{1}{2}}\overline{fg} = \bar{f}*\bar{g}.$$

Thus

$$\overline{G(mX)\exp(iMX)} = (2\pi)^{-\frac{3}{2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(mX')e^{-i\xi'X'}\,dX'\int_{-\infty}^{\infty} e^{iMX''}e^{-i(\xi-\xi')X''}\,dX''\,d\xi'$$

or, letting $mX' = x'$, $\xi' = m\rho$, $mX'' = -x''$, one has

$$\overline{G(mX)\exp(iMX)} = m^{-1}(2\pi)^{-\frac{3}{2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x')\mathrm{e}^{-i\rho x'}dx'\int_{-\infty}^{\infty}\mathrm{e}^{-ix''[\rho-(\xi-M)/m]}dx''d\rho.$$

Interchanging the order of integration and applying the Fourier integral theorem one finally obtains

$$\overline{G(mX)\exp(iMX)} = m^{-1}(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty} G(x')\mathrm{e}^{-i[(\xi-M)/m]x'}dx'$$

$$= m^{-1}\overline{G(x)}[(\xi - M)/m] = m^{-1}\overline{G(x)}\left[\frac{\xi/M - 1}{\beta}\right].$$

Inserting this into (16) and carrying out the $t'$ integral, one finds

$$A(\xi, t) = -\frac{\overline{g\gamma(mX)}k^{-1}\sin kt}{\cosh\xi h}$$

$$+\frac{\varepsilon g^2\overline{G(x)}\left[\frac{\xi/M-1}{\beta}\right]}{\beta\sigma\cosh\xi h}\left\{-\frac{\mathrm{e}^{-i\sigma t}}{\sigma^2-k^2}-\frac{\mathrm{e}^{ikt}}{2k(\sigma+k)}+\frac{\mathrm{e}^{-ikt}}{2k(\sigma-k)}\right\}.$$

The surface pattern is then given by, [cf. (14)]

$$\eta(X, t) = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}\frac{\overline{\gamma(mX)}\cos kt}{\cosh\xi h}\mathrm{e}^{i\xi X}d\xi$$

$$-\frac{i\varepsilon g}{(2\pi)^{\frac{1}{2}}\beta\sigma}\int_{-\infty}^{\infty}\frac{\overline{G(x)}\left[\frac{\xi/M-1}{\beta}\right]}{\cosh\xi h}$$

$$\times\left\{\frac{\sigma\mathrm{e}^{-i\sigma t}}{\sigma^2-k^2}-\frac{\mathrm{e}^{ikt}}{2(\sigma+k)}-\frac{\mathrm{e}^{-ikt}}{2(\sigma-k)}\right\}\mathrm{e}^{i\xi X}d\xi.$$

Going over to the variable $\zeta = \xi h$ and setting $T = (g/h)^{\frac{1}{2}}t$, one obtains

$$\eta(X, t) = h^{-1}(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}\frac{\overline{\gamma(mX)}\cos[(\zeta\tanh\zeta)^{\frac{1}{2}}T]}{\cosh\zeta}\mathrm{e}^{i\zeta X/h}d\zeta$$

$$-\frac{i\varepsilon(g/h)^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}\beta\sigma}\int_{-\infty}^{\infty}\frac{\overline{G(x)}\left[\frac{\zeta/Mh-1}{\beta}\right]}{\cosh\zeta}\left\{\frac{(h/g)^{\frac{1}{2}}\sigma\mathrm{e}^{-i\sigma t}}{Mh\tanh Mh-\zeta\tanh\zeta}\right.$$

$$-\frac{\mathrm{e}^{i(\zeta\tanh\zeta)^{\frac{1}{2}}T}}{2[(Mh\tanh Mh)^{\frac{1}{2}}+(\zeta\tanh\zeta)^{\frac{1}{2}}]}$$

$$\left.-\frac{\mathrm{e}^{-i(\zeta\tanh\zeta)^{\frac{1}{2}}T}}{2[(Mh\tanh Mh)^{\frac{1}{2}}-(\zeta\tanh\zeta)^{\frac{1}{2}}]}\right\}\mathrm{e}^{i\zeta X/h}d\zeta. \qquad (17)$$

Formula (17) gives the surface shape corresponding to the solution of the initial-value problem (12a–d).

**B-II. Determination of the steady-state motion – Approximation for shallow water**

One determines the steady-state wave motion by evaluating the limit of $\eta(X, t)$ in (17) as $T \to \infty$. We first define the function $(\zeta \tanh \zeta)^{\frac{1}{2}}$ as an analytic function in the neighborhood of the real axis by the formula $(\zeta \tanh \zeta)^{\frac{1}{2}} = \zeta(\tanh \zeta/\zeta)^{\frac{1}{2}}$. Since it is strictly increasing on the real axis one can apply the Riemann–Lebesgue lemma directly to the first integral in (17) to deduce that it is of order $O(T^{-1})$ as $T \to \infty$ for fixed $X$. To evaluate the limit in the second integral, we bring it through the brackets, deforming the path above the singularity $\zeta = -Mh$ and below the singularity $\zeta = Mh$ [cf. (21a)]. Integrating on this deformed path, $C$, insures that the contributions of the second and third terms in brackets go to zero for large $T$. In fact, one can return to a straight path through $\zeta = Mh$ in the integral with the second term in brackets and then on the straight sections of the resulting path, the integral with the second term is of order $O(T^{-1})$ and of negative exponential order on the deformed section above $\zeta = -Mh$ since there $\operatorname{Im}[(\zeta \tanh \zeta)^{\frac{1}{2}}] > 0$. The integral with the third term in brackets exhibits similar behavior and, therefore, for large $T$ one is left with the steady-state term:

$$\eta(X, t) = -\frac{i\varepsilon e^{-i\sigma t}}{(2\pi)^{\frac{1}{2}}\beta} \int_C \frac{\overline{G(x)\left[\dfrac{\zeta/Mh - 1}{\beta}\right]} e^{i\zeta X/h}}{Mh \tanh(Mh) \cosh \zeta - \zeta \sinh \zeta} d\zeta. \tag{18}$$

We now make the assumption that the water is shallow compared with the wavelength of the initial wave and moreover we assume that the depth is negligible compared with the variation rate of the bottom elevation:

$$\delta \ll 1, \quad \delta \ll \beta. \tag{19a, b}$$

To carry out the integral in (18) one needs to know the roots of the equation

$$\zeta \tanh \zeta = Mh \tanh Mh. \tag{20}$$

Eq. (20) has real and pure imaginary roots and no others. We evaluate them taking into account (19a). Letting $\zeta = \lambda + i\mu$, one obtains for the real roots

$$\zeta = \pm Mh. \tag{21a}$$

The pure imaginary roots satisfy the equation

$$\mu \tan \mu = -Mh \tanh Mh$$

and are

$$\zeta = \pm in\pi, \quad n = 1, 2, 3, \ldots . \tag{21b}$$

The derivative of the denominator in (18),

$$f(\zeta) = Mh \tanh(Mh) \cosh \zeta - \sinh \zeta,$$

at the above roots has the values

$$f'(\pm Mh) = \mp 2Mh, \tag{22}$$

$$f'(\pm in\pi) = \mp(-1)^n in\pi, \quad n = 1, 2, 3, \ldots . \tag{23}$$

Letting $X = x/m$, we write (18) in the form

$$\eta(X, t) = (2\pi i\beta)^{-1}\varepsilon e^{-i\sigma t} \int_C \int_{-\infty}^{\infty} \frac{G(x')e^{ix'/\beta}e^{i\zeta(x-x')/Mh\beta}}{f(\zeta)} d\zeta. \tag{24}$$

For $x > x'$, we form a closed path of $\zeta$ integration, $C_+$, by adding to the part of $C$ for which $|Re(\zeta)| \leqq R > 0$, the semicircular segment $\zeta = Re^{i\theta}$, $0 \leqq \theta \leqq \pi$. As $R \to \infty$ the integral on the semicircular segment goes to zero and one can thus evaluate (24) by the residue theorem. For $x < x'$, one adds the segment $\zeta = Re,^{i\theta}$ $-\pi \leqq \theta \leqq 0$. If one uses (21a, b), (22), (23), the result is:

$$\eta(X, t) = \beta^{-1}\varepsilon e^{-i\sigma t} \int_{-\infty}^{x} G(x')e^{ix'/\beta}$$

$$\cdot\left\{\frac{e^{i(x-x')/\beta}}{-2Mh} + \sum_1^\infty \frac{e^{-n\pi(x-x')/Mh\beta}}{-i(-1)^n n\pi}\right\} dx'$$

$$- \beta^{-1}\varepsilon e^{-i\sigma t} \int_{x}^{\infty} G(x')e^{ix'/\beta}$$

$$\cdot\left\{\frac{e^{-i(x-x')/\beta}}{2Mh} + \sum_1^\infty \frac{e^{n\pi(x-x')/Mh\beta}}{i(-1)^n n\pi}\right\} dx'.$$

Setting $x = mX$, and using (6a), (6d), (7a), one has

$$\eta(X, t) = -(2D)^{-1}C \exp[i(MX - \sigma t)] \int_{-\infty}^{mX} G(x')dx'$$

$$- (2D)^{-1}C \exp[-i(MX + \sigma t)] \int_{mX}^{\infty} G(x')e^{2ix'/\beta}dx'$$

$$+ (iC\delta^2/D)e^{-i\sigma t} \int_{-\infty}^{mX} G(x') \sum_1^\infty (-1)^n(n\pi)^{-1} \exp[-n\pi(mX - x')/Mh\beta + ix'/\beta]dx$$

$$+ (iC\delta^2/D)e^{-i\sigma t} \int_{mX}^{\infty} G(x') \sum_1^\infty (-1)^n(n\pi)^{-1} \exp[n\pi(mX - x')/Mh\beta + ix'/\beta]dx'.$$

The first two terms are of order $O(\varepsilon D^{-1})$ and the last two are of order $O[\varepsilon\delta^2(e^{-\pi/\beta} + \delta^2)D^{-1}]$. To see the latter, one subdivides the interval of integration in, for example, the third term into two at the point $x' = mX - \delta^2$ and applies straightforward integral estimates. We neglect the last two terms with respect to the first two by virtue of (19a). Using (9) and (11) one finally obtains the steady-state shallow-water surface shape corresponding to the solution of (2a–c), (5a, b):

$$\eta(X, t)/C = \exp[i(MX - \sigma t)]\left\{i - (1/2)\int_{-\infty}^{mX} [\gamma(x) - i\beta\gamma'(x)]\,dx\right\}$$

$$- (1/2)\exp[-i(MX + \sigma t)]\int_{mX}^{\infty} [\gamma(x) - i\beta\gamma'(x)]\exp(2ix/\beta)\,dx. \tag{25}$$

**B-III Properties of the steady-state motion**

The expression (25) gives a decomposition of the steady state surface shape into forward and backward progressing waves whose amplitude and phase exhibit a slow spatial modulation due to the variation in depth. Since the forward wave coincides with the wave (4a) at $X = -\infty$ and the amplitude of the backward wave vanishes at $X = \infty$, we shall interpret (4a) as an incoming wave (from $X = -\infty$), the forward wave as a transmitted wave, and the backward wave as a reflected wave.

*Reflection coefficient*

At this point our only assumptions on $\gamma(x)$ have been that $\gamma$ and $\gamma'$ exist and are absolutely integrable. We now study the magnitude of the reflected wave in (25) when the $N$th derivative, $\gamma^{(N)}(x)$, $N \geqq 0$, exists and is piecewise continuous with at most jump discontinuities. We find that, when $\beta$ is small, the main contribution to the reflection coefficient comes from the jumps in $\gamma^{(N)}$ and, moreover, that for all $\beta < 1$ the magnitude of the reflection coefficient decreases as $N \to \infty$. In fact, we show that when $\gamma \in C_\infty$ and of compact support then there is no reflection whatsoever.

We first investigate the reflection when the bottom profile is piecewise continuous. Let $\gamma(x)$ have a jump discontinuity at $x = x_0$ with the magnitude $J_0(x_0) = \gamma(x_0 + 0) - \gamma(x_0 - 0)$. Assuming that $\gamma'$ is bounded on $(-\infty, x_0]$ and $[x_0, \infty)$, we set $X < X_0$ and integrate by parts in the second term of (25) obtaining

$$\eta(X, t)/C = \exp[i(MX - \sigma t)]\left\{i - (1/2)\int_{-\infty}^{mX} \gamma(x)\,dx\right\}$$

$$+ (1/2)\exp[-i(MX + \sigma t)]\left\{\int_{mX}^{\infty} \gamma(x)\exp(2ix/\beta)\,dx - i\beta J_0(x_0)\exp(2iMX_0)\right\}. \tag{26}$$

Following Kreisel [2] we define the left-hand reflection coefficient as the ratio of the absolute values of reflected-to incoming-wave amplitudes at $X = -\infty$:

$$R = (1/2)\left|\int_{-\infty}^{\infty} \gamma(x)\exp(2ix/\beta)\,dx - i\beta J_0(x_0)\exp(2iMX_0)\right|. \tag{27}$$

Formula (27) includes a result of Kreisel [2]. By a quite different method he has obtained the following reflection coefficient for reflection of waves of wave number $M$ by the continuous, non-slowly-varying bottom profile $y = -h + D\gamma(X)$ in the limit as $Mh \to 0$:

$$R_0 = (DM/h)\left|\int_{-\infty}^{\infty} \gamma(X) \exp(2iMX)\, dX\right|. \tag{27'}$$

Formula (27′) is, in fact, a special case of (27) since it is obtained from the latter by setting $m = 1$, $J_0(x_0) = 0$.

We now investigate the consequences of taking $\beta$ to be small. We first show that, then, the main contribution to $R$ in (27) will arise from the discontinuity in $\gamma$. To see this, we integrate by parts in (27), obtaining

$$R = (\beta/4)\left|J_0(x_0)\exp(2iMX_0) - \int_{-\infty}^{\infty} \gamma'(x)\exp(2ix/\beta)\, dx\right|$$

or, applying the Riemann-Lebesgue lemma to the integral, one gets

$$R = (\beta/4)\,|J_0(x_0)\exp(2iMX_0) + O(\beta)|.$$

This result is a particular case of the following general theorem according to which a bottom profile with a jump discontinuity in the $N$th derivative will give rise to a reflected wave of magnitude $O[(\beta/2)^{N+1}] = O[(D/2h)(\beta/2)^N]$.

*Theorem 1. Let* $\gamma^{(N)}(x)$, $N \geqq 0$, *exist and be continuous except for a jump discontinuity at* $x = x_0$ *with the magnitude* $J_N(x_0) = \gamma^{(N)}(x_0 + 0) - \gamma^{(N)}(x_0 - 0)$, *and let* $\gamma^{(n)}(x)$, $0 \leqq n \leqq N + 1$, *exist and be absolutely integrable. Then the reflected wave in* (25) *has a reflection coefficient given by*

$$R_N = \tfrac{1}{2}(\beta/2)^{N+1}\,|J_N(x_0)\exp(2iMX_0) + O(\beta)|.$$

*Proof:* The case $N = 0$ is shown above. For $N > 0$, we take $J_0(x_0) = 0$ in (26), set $X < X_0$, and integrate repeatedly by parts obtaining

$$\eta(X,t)/C = \exp[i(MX - \sigma t)]\left\{i - (1/2)\int_{-\infty}^{mX} \gamma(x)\, dx + (1/2)\sum_{n=0}^{N} (i\beta/2)^{n+1}\gamma^{(n)}(mX)\right\}$$

$$+ \exp[-i(MX + \sigma t)](1/2)(i\beta/2)^{N+1}[J_N(x_0)\exp(2iMX_0) + O(\beta)],$$

from which the result follows.

*Transmitted wave—phase shift and local wave number*

We now investigate the properties of the transmitted wave in (25). In particular, we obtain expressions for the phase shift and the local wave number. We compare them with those predicted by the ray theory for the purpose of determining the region of validity of that theory, applied to the present problem, in terms of the parameter $\beta$.

A comparison with the ray theory is most easily made if $\gamma$ is chosen in such a way that there is no reflection. According to Theorem 1 this will be guaranteed by taking $\gamma \in C_\infty$ and of compact support. The transmitted wave then takes the real form

$$-\eta(X, t)/C = [1 + (\beta/2) \sum_0^\infty (-1)^n (\beta/2)^{2n} \gamma^{(2n)}(mX)] \sin(MX - \sigma t)$$

$$+ (1/2)\left[\int_{-\infty}^{mX} \gamma(x)\,dx + (\beta/2)^2 \sum_0^\infty (-1)^n (\beta/2)^{2n} \gamma^{(2n+1)}(mX)\right] \cos(MX - \sigma t) \qquad (28)$$

where the derivatives of $\gamma$ are assumed be such that the series are convergent.

If (28) is brought into the form

$$-\eta(X, t)/C = a(mX) \sin[MX - \sigma t + p(mX)], \qquad (29)$$

then $p(mX)$ represents the phase shift in the transmitted wave from $-\infty$ to $X$; it can be calculated in terms of $\beta$ to any desired accuracy. The leading term is

$$p(mX) = (1/2) \int_{-\infty}^{mX} \gamma(x)\,dx + O(\beta^2). \qquad (30)$$

$a(mX)$ is the local amplitude and can also be calculated to any desired accuracy.

The spatial variation in the phase is associated with a corresponding variation in the wave number. This "local" wave number $k$ is defined to be the partial derivative with respect to $X$ of the argument of the sine function in (29), i.e. $k(mX) = M + mp'(mX)$ or, from (30),

$$k(mX) = M[1 + (\beta/2)\gamma(mX) + O(\beta^3)]. \qquad (31)$$

One notes that if the bottom elevation is sufficiently slowly varying, that is neglecting $O(\beta^3)$, the wave number is determined by the bottom elevation alone and is not influenced by its derivatives. This important fact is used as the basis for the ray theory. More precisely, the ray theory rests on the assumption that, if the bottom elevation is sufficiently slowly varying, the local wave number can be defined with sufficient accuracy by (3b) considering $M$ and $h$ to be variable. For shallow water, (19a), this wave number is given by

$$\bar{k}(mX) = M/[1 - D\gamma(mX)/h]^{\frac{1}{2}} = M/[1 - \beta\gamma(mX)]^{\frac{1}{2}}. \qquad (32)$$

The result (31) enables one to verify this assumption and, moreover, to calculate the error incurred in using (32). In fact, comparing (32) and (31), the error incurred will be of order $O(\beta^2) = O(\beta D/h)$.

The significance of $\bar{k}$ in the ray theory is that the phase function is obtained from it by integration. We now determine the error in the phase shift predicted by the ray theory which is induced by the error in $\bar{k}$. If one denotes the phase function of the transmitted wave (29) by $P(x) = S(x)/m - \sigma t$, $S$ can be shown to satisfy, [5],

$$S' = \bar{k}(x).$$

If one requires that $S \to Mx$ as $x \to -\infty$, this has the solution

$$S(x) = Mx + \int_{-\infty}^{x} [\bar{k}(x') - M]\,dx',$$

which yields

$$P(x) = MX - \sigma t + (1/2)\int_{-\infty}^{mX} \gamma(x)dx + O(\beta).$$

Comparing with (29), (30), the error in the phase shift predicted by the ray theory is of order $O(\beta)$.

## PART C. THE THREE-DIMENSIONAL PROBLEM

### C-I. Solution of the initial-value problem

In this part we consider $\gamma(x, z)$ as defined in Part A with the additional assumption that it is of compact support. To obtain a solution of (12a–d) we utilize the Fourier transformation with respect to $X$ and $Z$:

$$\bar{\psi}(\xi, y, \zeta, t) = (2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi(X, y, Z, t)e^{-i\xi X - i\zeta Z}dX\,dZ.$$

The initial value problem (12a–d) then becomes

$$\bar{\psi}(\xi, y, \zeta, 0) = \bar{\psi}_t(\xi, y, \zeta, 0) = 0, \quad -h \leqq y \leqq 0; \tag{33a}$$

$$-\xi^2\bar{\psi} + \bar{\psi}_{yy} - \zeta^2\bar{\psi} = 0, \quad -h \leqq y \leqq 0; \tag{33b}$$

$$\bar{\psi}_{tt} + g\bar{\psi}_y = 0, \quad y = 0; \tag{33c}$$

$$\bar{\psi}_y = a'\bar{\gamma} - (\varepsilon gMa/\sigma)\exp(-i\sigma t)\overline{G(mX, mZ)\exp(iMX)}, \quad y = -h. \tag{33d}$$

From (33b),

$$\bar{\psi}(\xi, y, \zeta, t) = A(\xi, \zeta, t)\cosh(\rho y) + B(\xi, \zeta, t)\sinh(\rho y), \tag{34}$$

where $\rho = (\xi^2 + \zeta^2)^{\frac{1}{2}}$. Substituting (34) into (33c, d) and eliminating $B$ one gets an ordinary differential equation for $A$ in the variable $t$:

$$A_{tt} + (g\rho\tanh\rho h)A$$
$$= [-g\bar{\gamma}a' + (\varepsilon g^2Ma/\sigma)\exp(-i\sigma t)\overline{G(mX, mZ)\exp(iMX)}]/\cosh\rho h, \quad t > 0. \tag{35a}$$

From (33a) and (34) one deduces the initial conditions

$$A(\xi, \zeta, 0) = A_t(\xi, \zeta, 0) = 0. \tag{35b}$$

If one denotes $k(\xi, \zeta) = (g\rho\tanh\rho h)^{\frac{1}{2}}$, the initial value problem (35a, b) has the solution

$$A(\xi, \zeta, t) = \int_0^t \frac{k^{-1}\sin[k(t - t')]}{\cosh\rho h}\{-g\bar{\gamma}a'(t')$$
$$+ (\varepsilon g^2Ma/\sigma)\exp(-i\sigma t')\overline{G(mX, mZ)\exp(iMX)}\}\,dt'.$$

We integrate by parts with the first term in brackets, using (1), to obtain

$$A(\xi,\zeta,t)=\frac{-g\bar{\gamma}}{\cosh\rho h}\left\{k^{-1}\sin[k(t-\alpha)]+\int_0^{\alpha}\cos[k(t-t')]a(t')dt'\right\}$$

$$+\frac{(\varepsilon g^2M/\sigma)\overline{G(mX,mZ)\exp(iMX)}}{\cosh\rho h}\int_0^t k^{-1}\sin[k(t-t')]\exp(-i\sigma t')a(t')dt'.$$

We now take $\alpha$ in (1) to be of the order of magnitude of the quantities that we have heretofore neglected and write

$$A(\xi,\zeta,t)=-\frac{g\bar{\gamma}k^{-1}\sin kt}{\cosh\rho h}$$

$$+\frac{(\varepsilon g^2M/\sigma)\overline{G(mX,mZ)\exp(iMX)}}{\cosh\rho h}\int_0^t k^{-1}\sin[k(t-t')]\exp(-i\sigma t')dt'. \qquad (36)$$

To evaluate the transform of a product which appears in (36), we make use of the convolution theorem in the form

$$2\pi\overline{fg}=\bar{f}*\bar{g}.$$

Thus

$$\overline{G(mX,mZ)\exp(iMX)}=(2\pi)^{-3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(mX',mZ')e^{-i\xi'X'-i\zeta'Z'}dX'dZ'$$

$$\cdot\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp(iMX'')e^{-i(\xi-\xi')X''-i(\zeta-\zeta')Z''}dX''dZ''d\xi'd\zeta',$$

or letting $x'=mX'$, $z'=mZ'$, $\xi'=m\nu$, $\zeta'=m\mu$, $x''=mX''$, $z''=mZ''$, one has

$$\overline{G(mX,mZ)\exp(iMX)}=m^{-2}(2\pi)^{-3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(x',z')e^{-i\nu x'-i\mu z'}dx'dz'$$

$$\cdot\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{ix''(\nu-\xi/m+M/m)+iz''(\mu-\zeta/m)}dx''dz''d\nu d\mu.$$

Interchanging the order of integration and applying the Fourier integral theorem one finally obtains

$$\overline{G(mX,mZ)\exp(iMX)}=\frac{1}{m^2 2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(x',z')e^{-i[(\xi-M)/m]x'-i(\zeta/m)z'}dx'dz'$$

$$=m^{-2}\overline{G(x,z)}\left[\frac{\xi/M-1}{\beta},\frac{\zeta/M}{\beta}\right].$$

Here, the bar over $G(x,z)$ denotes the transform with respect to $x$ and $z$ and the square brackets contain the arguments of the transformed function.

Inserting this into (36) and carrying out the $t'$ integral, one obtains

$$A(\xi, \zeta, t) = -\frac{g\bar{\gamma}k^{-1}\sin kt}{\cosh ph}$$

$$+\frac{\varepsilon g^2\overline{G(x, z)}\left[\dfrac{\xi/M - 1}{\beta}, \dfrac{\zeta/M}{\beta}\right]}{\sigma\beta^2 M\cosh ph}\left\{-\frac{e^{-i\sigma t}}{\sigma^2 - k^2} - \frac{e^{ikt}}{2k(\sigma + k)} - \frac{e^{-ikt}}{2k(\sigma - k)}\right\}.$$

The surface pattern corresponding to the potential $\psi$ is then given by, [cf. (34)],

$$\eta(X, Z, t) = -g^{-1}\psi_t(X, 0, Z, t)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\overline{\gamma(mX)}\cos kt}{\cosh ph}\,e^{i\xi X + i\zeta Z}\,d\xi\,d\zeta$$

$$-\frac{i\varepsilon g}{2\pi\sigma\beta^2 M}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\overline{G(x, z)}\left[\dfrac{\xi/M - 1}{\beta}, \dfrac{\zeta/M}{\beta}\right]}{\cosh ph}\left\{\frac{\sigma e^{-i\sigma t}}{\sigma^2 - k^2}\right.$$

$$\left.-\frac{e^{ikt}}{2(\sigma + k)} - \frac{e^{-ikt}}{2(\sigma - k)}\right\}e^{i\xi X + i\zeta Z}\,d\xi\,d\zeta.$$

We make the following change of coordinates

$$\xi h = u\cos\phi,\ \zeta h = u\sin\phi, \qquad X = R\cos\theta,\ Z = R\sin\theta.$$

Setting $r = mR$ and $T = (g/h)^{\frac{1}{2}}t$, one gets

$$\eta(R, \theta, t) = \frac{1}{2\pi h^2}\int_{-\pi}^{\pi}\int_{0}^{\infty}\frac{\bar{\gamma}\cos[(u\tanh u)^{\frac{1}{2}}T]}{\cosh u}\,e^{iu(R/h)\cos(\phi - \theta)}\,u\,du\,d\phi$$

$$-\frac{i\varepsilon(g/h)^{\frac{1}{2}}}{2\pi\sigma\beta^2\delta^2}\int_{-\pi}^{\pi}\int_{0}^{\infty}\frac{\overline{G(r, \theta)}\left[\dfrac{u\cos\phi/Mh - 1}{\beta}, \dfrac{u\sin\phi/Mh}{\beta}\right]}{\cosh u}$$

$$\times\left\{\frac{(h/g)^{\frac{1}{2}}\sigma e^{-i\sigma t}}{Mh\tanh Mh - u\tanh u} - \frac{e^{i(u\tanh u)^{\frac{1}{2}}T}}{2[(Mh\tanh Mh)^{\frac{1}{2}} + (u\tanh u)^{\frac{1}{2}}]}\right.$$

$$\left.-\frac{e^{-i(u\tanh u)^{\frac{1}{2}}T}}{2[(Mh\tanh Mh)^{\frac{1}{2}} - (u\tanh u)^{\frac{1}{2}}]}\right\}e^{iu(R/h)\cos(\phi - \theta)}\,u\,du\,d\phi. \qquad (37)$$

Formula (37) is the surface shape corresponding to the solution of the initial-value problem (12a–d).

**C-II. Determination of the steady-state motion – Approximation for shallow water**

One determines the steady-state wave motion by evaluating the limit of $\eta(R, \theta, t)$ in (37) as $T \to \infty$. We note that the function $(u\tanh u)^{\frac{1}{2}}$ is analytic in a neighborhood of the positive real axis. Moreover, on the real axis it is strictly increasing and one can apply the

Riemann-Lebesgue lemma directly to the first integral in (37) to deduce that it is of order $O(T^{-1})$ as $T \to \infty$ for fixed $R$, $\theta$.

In the second term of (37) we deform the path of $u$-integration below the singularity $u = Mh$ into, say, a semicircle and then bring the $u$-integral into the brackets. Integrating on this deformed path, $C$, insures that the contributions of the second and third terms in brackets go to zero for large $T$. In fact, one can return to a straight path through $u = Mh$ in the integral with the second term in brackets, and its contribution will be $O(T^{-1})$ by the Riemann-Lebesgue lemma. As for the integral with the third term, it will be $O(T^{-1})$ on the straight sections of $C$ and of negative exponential order in $T$ on the deformed section below $u = Mh$ since there $\mathrm{Im}[(u \tanh u)^{\frac{1}{2}}] < 0$. Thus, as $T \to \infty$, one is left with the steady-state term

$$\eta(R, \theta, t) = -\frac{i\varepsilon e^{-i\sigma t}}{2\pi\beta^2\delta^2} \int_{-\pi}^{\pi} \int_C \frac{\overline{G(r, \theta)}\left[\dfrac{u\cos\phi/Mh - 1}{\beta}, \dfrac{u \sin\phi/Mh}{\beta}\right]}{Mh \tanh(Mh)\cosh u - u \sinh u} \times e^{iu(R/h)\cos(\phi-\theta)} u\, du\, d\phi.$$

We write this in the form

$$\eta(R, \theta, t) = -\frac{i\varepsilon e^{-i\sigma t}}{(2\pi)^2\beta^2\delta^2} \int_C \int_{-\pi}^{\pi} \int_0^{\infty} \int_{-\pi}^{\pi} \frac{G(r', \theta') e^{iu[r\cos(\phi-\theta) - r'\cos(\phi-\theta')]/Mh\beta}}{Mh \tanh(Mh)\cosh u - u \sinh u} u\, du\, d\phi \times e^{ir'\cos\theta'/\beta} r'\, dr'\, d\theta'. \tag{38}$$

We now assume (19a, b) as well as assuming that $\beta$ is small. To evaluate (38) we first consider the case:

$$r\cos(\phi - \theta) - r'\cos(\phi - \theta') < 0. \tag{39a}$$

On the straight part of $C$ the $u$ integral will be $O(Mh\beta) = O(\beta\delta^2)$ by the Riemann–Lebesgue lemma. On the semicircle below $u = Mh$ one has $\mathrm{Im}(u) < 0$. Now since $u = Mh$ is close to the endpoint $u = 0$ of the path $C$, [cf. (19a)], the semicircle will have at most radius $Mh$. Hence the quantity $u/Mh$ is $O(1)$ on the semicircle. Therefore the $u$ integral on the semicircle will be of negative exponential order in $1/\beta$. The contribution of the $u$ integral to $\eta$ in the case (39a) is thus $O[(\varepsilon e^{-k/\beta} + \varepsilon\beta\delta^2)/D]$, which we neglect on account of (19a) and the assumption that $\beta$ is small.

Taking the case

$$r\cos(\phi - \theta) - r'\cos(\phi - \theta') > 0, \tag{39b}$$

the $u$ integral will again be $O(\beta\delta^2)$ on the straight part of $C$. On the semicircle, however, the $u$ integral is no longer of negative exponential order in $1/\beta$ and in fact its limit as $\beta \to 0$ is not clear. To remedy this we push the semicircle up through $u = Mh$ into a semicircle running above it. On this upper semicircle one has $\mathrm{Im}(u) > 0$ and the $u$ integral will again be of negative exponential order in $1/\beta$. The dominant contribution is that due to the residue at $u = Mh$ and one gets in view of (6b), (7b),

$$\eta(R, \theta, t) = -\frac{C e^{-i\sigma t}}{4\pi D} \int_0^{\infty} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} G(r', \theta') e^{i\beta^{-1}[r\cos(\phi-\theta) - r'\cos(\phi-\theta') + r'\cos\theta']} r'\, dr'\, d\theta'\, d\phi \tag{40}$$

subject to (39b). Formula (40) gives the steady-state shallow-water surface shape corresponding to the solution of the initial-value problem (12a–d).

**C-III. Asymptotic development of the steady-state motion for small bottom variation rate**

We now obtain the asymptotic development of (40) as $\beta$ tends to zero. We shall restrict our investigation to those bottom configurations for which $\gamma_{zz}$ exists and is continuous for each $x$ and $z$. Our main tool will be the method of stationary phase applied to a double integral [11]. For convenience we return to the rectangular coordinates

$$x' = r'\cos\theta',\ z' = r'\sin\theta', \qquad x = r\cos\theta,\ z = r\sin\theta;$$

then (39b) and (40) become

$$\eta(X, Z, t) = -\frac{C\mathrm{e}^{-i\sigma t}}{4\pi D}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\pi/2}^{3\pi/2} G(x'z')\mathrm{e}^{i\beta^{-1}[(x-x')\cos\phi+(z-z')\sin\phi+x']}dx'\,dz'\,d\phi, \tag{41}$$

$$(x - x')\cos\phi + (z - z')\sin\phi > 0. \tag{42}$$

In (41) we have shifted the region of $\phi$ integration so that the stationary points will fall in the interior of the region of integration, [cf. (43a, b)].

To obtain the asymptotic development of (41), (42), we apply the method of stationary phase to the double $\phi - z'$ integral, the variables $x$ and $x'$ being considered as parameters. The stationary points of the phase function

$$p(\phi, z') = (x - x')\cos\phi + (z - z')\sin\phi + x'$$

are given by the roots of the system

$$p_\phi = -(x - x')\sin\phi + (z - z')\cos\phi = 0, \qquad p_{z'} = -\sin\phi = 0,$$

subject to (42). One obtains the two stationary points:

$$\phi = 0, \qquad z = z', \qquad (x > x') \tag{43a}$$

$$\phi = \pi, \qquad z = z', \qquad (x < x'). \tag{43b}$$

These stationary points are both of first order since the expression $|p_{\phi\phi}p_{z'z'} - p^2_{\phi z'}|$ equals 1 at each of them.

Following Chako [11] one uses a neutralizer function to show that the principal contribution to (41) as $\beta \to 0$ arises from small neighborhoods of the stationary points. Moreover, Chako has shown that the difference, $R$, between this principal contribution and the actual contribution is of order $O(\beta^N)$ where $N$ is the order of differentiability of $G(x', z')$ with respect to $z'$. From the above assumption that $\gamma_{zz}$ is continuous and from (11) one has $N = 2$ and hence $R = O(\beta^2)$. We henceforth neglect $R$ since we are ultimately interested only in the leading term of the principal contribution which will be of order $O(\beta)$. The principal contribution of (41), (42) will appear as follows:

$$\eta(X, Z, t) = \eta_1(X, Z, t) + \eta_2(X, Z, t),$$

where $\eta_1$ is the restriction of (41) to a small square $\phi - z'$ neighborhood about (43a) and $\eta_2$ is its restriction to a small square neighborhood about (43b).

The basic approach to the asymptotic evaluation of $\eta_1$, $\eta_2$ is to reduce the double $\phi - z'$ integral to a sum of products of single integrals since for the latter the method of stationary phase is well known. For this purpose we use Taylor's theorem to write $p(\phi, z')$ in the neighborhood of (43a) in the form

$$p(\phi, z') = x + (x' - x)\phi^2/2 - (z - z')\phi + (x - x')\phi^4/4! + (z' - z)\phi^3/3!$$
$$+ \left(\phi \frac{\partial}{\partial\phi} + (z' - z)\frac{\partial}{\partial z}\right)^5 p(\xi\phi, z + \xi(z' - z))/5! \tag{44}$$

where $0 < \xi < 1$. Here, the derivatives of $p$ are assumed taken before $\xi$ is substituted.

We eliminate the cross product in the quadratic terms of $p$ by introducing the affine transformation

$$\phi = u\cos\alpha - v\sin\alpha, \qquad z' - z = u\sin\alpha + v\cos\alpha, \tag{45}$$

where $\alpha$ depends on $x - x'$ and is given by

$$\cot 2\alpha = (x - x')/2.$$

From (43a) one has $x > x'$ hence $0 < \alpha < \pi/4$ and moreover $\alpha \to 0_+$ as $x' \to -\infty$. Under this transformation the phase function (44) becomes

$$p(u, v) = x - u^2(\cot\alpha)/2 + v^2(\tan\alpha)/2 + H_4(u, v) + R_4(u, v), \tag{46}$$

where $H_4$ is a homogeneous polynomial in $u$ and $v$ of degree 4:

$$H_4(u, v) = \sum_{n=0}^{4} a_{nm}u^n v^m, \tag{47}$$

where the $a_{nm}$ depend on $x - x'$ and

$$R_4(u, v) = \left(u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\right)^5 q(u, v)/5!, \tag{48}$$

where $q(u, v)$ is continuous in $u$ and $v$ together with its fifth derivatives.

In addition, we use the mean value theorem to write $G(x', z')$ near $z' = z$ in the form

$$G(x', z') = G(x', z) + G_{z'}(x', \zeta(z'))(z' - z), \tag{49}$$

where $z < \zeta(z') < z'$ when $z' > z$ and similarly for $z' < z$. Expressing $z'$ in terms of $u$ and $v$ according to (45), we write $z' = z'(u, v)$ and $G(x', z') = G(x', u, v,)$ and (49) becomes

$$G(x', u, v) = G(x', z) + G_{z'}(x', \zeta[z'(u, v)])(u\sin\alpha + v\cos\alpha). \tag{50}$$

We substitute the expressions (46) and (50) for $p$ and $G$ into $\eta_1$, where the double $u, v$ integral is taken over a small square neighborhood, $N_b$, of the origin of side $2b$. We do not include the neutralizer function explicitly since it has no effect on the final asymptotic expression:

$$\eta_1(X, Z, t) = -\frac{C\,e^{i(MX-\sigma t)}}{4\pi D}\left\{\int_{-\infty}^{\infty} G(x', z)\iint_{N_b} e^{i\beta^{-1}[-(u^2/2)\cot\alpha+(v^2/2)\tan\alpha+H_4+R_4]}du\,dv\,dx'\right.$$

$$+\int_{-\infty}^{\infty}\iint_{N_b} G_{z'}(x', \zeta[z'(u, v)])(u\sin\alpha + v\cos\alpha)$$

$$\left.\cdot e^{i\beta^{-1}[-(u^2/2)\cot\alpha+(v^2/2)\tan\alpha+H_4+R_4]}du\,dv\,dx'\right\}. \quad (51)$$

We again use the mean-value theorem to write the exponential function which appears in the integrand in the form

$$e^{i\beta^{-1}[-(u^2/2)\cot\alpha+(v^2/2)\tan\alpha+H_4+R_4]} = e^{i\beta^{-1}[-(u^2/2)\cot\alpha+(v^2/2)\tan\alpha]}$$

$$\cdot[1 + i\beta^{-1}(H_4 + R_4)e^{i\theta\beta^{-1}(H_4+R_4)}]$$

where $0 < \theta < 1$. Inserting this in (51) one obtains after rearranging

$$\eta_1(X, Z, t) = -\frac{C\,e^{i(MX-\sigma t)}}{4\pi D}\left\{\int_{-\infty}^{\infty} G(x', z)\int_{-b}^{b} e^{-i\beta^{-1}(u^2/2)\cot\alpha}du\int_{-b}^{b} e^{i\beta^{-1}(v^2/2)\tan\alpha}dv\,dx'\right.$$

$$+\int_{-\infty}^{\infty}\iint_{N_b} G_{z'}(x', \zeta[z'(u, v)])(u\sin\alpha + v\cos\alpha)e^{i\beta^{-1}[-(u^2/2)\cot\alpha+(v^2/2)\tan\alpha]}du\,dv\,dx'$$

$$+ i\beta^{-1}\int_{-\infty}^{\infty} G(x', z)\iint_{N_b}(H_4 + R_4)e^{i\beta^{-1}[-(u^2/2)\cot\alpha+(v^2/2)\tan\alpha+\theta(H_4+R_4)]}du\,dv\,dx'$$

$$+ i\beta^{-1}\int_{-\infty}^{\infty}\iint_{N_b} G_{z'}(x', \zeta)(u\sin\alpha + v\cos\alpha)(H_4 + R_4)$$

$$\left.\cdot e^{i\beta^{-1}[-(u^2/2)\cot\alpha+(v^2/2)\tan\alpha+\theta(H_4+R_4)]}du\,dv\,dx'\right\}$$

$$= -\frac{C\,e^{i(MX-\sigma t)}}{4\pi D}[I_1 + I_2 + I_3 + I_4].$$

We now evaluate asymptotically, as $\beta \to 0$, each of the integrals $I_i$. We show that $I_1$ yields the dominant contribution, of order $O(\beta)$, and that $I_2$, $I_3$ and $I_4$ are each of order $O(\beta^2)$, at least.

In $I_1$ we let

$$s = u(\cot\alpha)^{\frac{1}{2}}, \quad t = v(\tan\alpha)^{\frac{1}{2}}, \quad (52)$$

obtaining

$$I_1 = \int_{-\infty}^{\infty} G(x', z)\int_{s(-b)}^{s(b)} e^{-i\beta^{-1}s^2/2}ds\int_{t(-b)}^{t(b)} e^{i\beta^{-1}t^2/2}dt\,dx'.$$

A direct application of the method of stationary phase to each of the $s$ and $t$ integrals yields, since $x > x'$ for stationary point (43a),

$$I_1 = 2\pi\beta \int_{-\infty}^{x} G(x', z)dx', \qquad \beta \to 0.$$

In $I_2$ we use (52) and bring in the $s$ and $t$ integrals to obtain

$$I_2 = \int_{-\infty}^{\infty} \frac{\sin\alpha}{(\cot\alpha)^{\frac{1}{2}}} \int_{s(-b)}^{s(b)} s\, e^{-i\beta^{-1}s^2/2} \int_{t(-b)}^{t(b)} G_{z'}(x', \zeta[z'(s,t)])e^{i\beta^{-1}t^2/2} ds\, dt\, dx'$$

$$+ \int_{-\infty}^{\infty} \frac{\cos\alpha}{(\tan\alpha)^{\frac{1}{2}}} \int_{s(-b)}^{s(b)} e^{-i\beta^{-1}s^2/2} \int_{t(-b)}^{t(b)} G_{z'}(x', \zeta[z'(s,t)])t\, e^{i\beta^{-1}t^2/2} ds\, dt\, dx'$$

$$= J_1 + J_2.$$

In $J_1$ we first apply stationary phase to the $t$-integral

$$J_1 = (2\pi\beta)^{\frac{1}{2}} e^{i\pi/4} \int_{-\infty}^{x} \frac{\sin\alpha}{(\cot\alpha)^{\frac{1}{2}}} \int_{s(-b)}^{s(b)} G_{z'}(x', \zeta[z'(s,0)])s\, e^{-i\beta^{-1}s^2/2} ds\, dx', \qquad \beta \to 0.$$

Now we integrate by parts in the $s$-integral. The boundary terms drop out due to the neutralizer:

$$J_1 = (2\pi)^{\frac{1}{2}} i\beta^{\frac{3}{2}} e^{i\pi/4} \int_{-\infty}^{x} \frac{\sin^2\alpha}{\cot\alpha} \int_{s(-b)}^{s(b)} G_{z'z'}(x', \zeta[z'(s,0)])\zeta'[z'(s,0)]\, e^{-i\beta^{-1}s^2/2} ds\, dx'.$$

Now the integrand of the $s$-integral is continuous in the region of $s$-integration under our assumption that $G_{z'z'}$ is continuous there. To see this one differentiates (49) with respect to $z'$, obtaining

$$G_{z'z'}(x', \zeta(z'))\zeta'(z') = \frac{G_{z'}(x', z') - G_{z'}(x', \zeta(z'))}{z' - z}$$

$$= G_{z'z'}(x', \chi) \frac{z' - \zeta(z')}{z' - z},$$

where $z < \zeta < \chi < z'$ for $z' > z$ and similarly for $z' < z$. This function in continuous in $z'$ in a neighborhood of $z$ and since $z'$ is a continuous function of $s$, the result follows. Since the integrand is continuous one can apply the method of stationary phase to the $s$ integral and the main contribution will arise from the stationary point. We obtain

$$J_1 = 2\pi i\beta^2 \int_{-\infty}^{x} \frac{\sin^2\alpha}{\cot\alpha} G_{z'z'}(x', z)\zeta'(z)dx', \qquad \beta \to 0.$$

Here we have written

$$G_{z'z'}(x', z)\zeta'(z) = \lim_{z' \to z} G_{z'z'}(x', \zeta(z'))\zeta'(z'),$$

where the limit exists by the above argument. Similarly

$$J_2 = -2\pi i\beta^2 \int_{-\infty}^{x} \frac{\cos^2 \alpha}{\tan \alpha} G_{z'z'}(x', z)\zeta'(z)dx', \qquad \beta \to 0.$$

The $x'$ integral in $J_2$ is convergent even though $\tan \alpha \to 0$ as $x' \to -\infty$ since we have assumed that $G$ is of compact support. Thus $I_2 = O(\beta^2)$.

We now show that $I_3$ and $I_4$ are also at least of order $O(\beta^2)$. To do this we simply let $u = U\beta^{\frac{1}{2}}$ and $v = V\beta^{\frac{1}{2}}$ and note from (47) that $H_4$ is a homogeneous polynomial in $u$ and $v$ of degree four and from (48) that $R_4$ consists of terms that are homogeneous in $u$ and $v$ of degree five except that each is multiplied by a continuous function of $u$ and $v$. Then taking $b = \beta^{\frac{1}{2}}$,

$$\begin{aligned} I_3 &= i\beta^2 \int_{-\infty}^{\infty} G(x', z) \iint_{N_1} [H_4(U, V) + \beta^{\frac{1}{2}} S_4(U, V)] \\ &\quad \cdot e^{i[-(U^2/2)\cot\alpha + (V^2/2)\tan\alpha + \beta\theta(H_4(U,V) + \beta^{\frac{1}{2}} S_4(U,V))]} dU\, dV\, dx' \\ &= O(\beta^2). \end{aligned}$$

Here $S_4(U, V) = \beta^{-\frac{5}{2}} R(U\beta^{\frac{1}{2}}, V\beta^{\frac{1}{2}})$ is of order unity. Similarly $I_4 = O(\beta^{\frac{5}{2}})$.

Therefore, the dominant contribution to $\eta_1$ arises from $I_1$ and (51) becomes

$$\eta_1(X, Z, t) = -C\beta(2D)^{-1} e^{i(MX - \sigma t)} \int_{-\infty}^{mX} G(x', z)dx' + O(\beta^2),$$

or, from (11),

$$\eta_1(X, Z, t) = -C\beta(2D)^{-1} e^{i(MX - \sigma t)} \int_{-\infty}^{mX} \gamma(x', z)dx' + O(\beta^2).$$

The asymptotic evaluation of $\eta_2$, that is, the principal contribution of (43b), proceeds along quite similar lines. The major difference in the result is the presence of the factor $e^{2ix'/\beta}$ in each $x'$ integral. This has the effect of raising the order of magnitude of each term by one due to the Riemann–Lebesgue lemma. The leading term is

$$\begin{aligned} \eta_2(X, Z, t) &= -C\beta(2D)^{-1} e^{-i(MX + \sigma t)} \int_{mX}^{\infty} G(x', z) e^{2ix'/\beta} dx' + O(\beta^3) \\ &= O(\beta^2). \end{aligned}$$

Hence the dominant contribution, of order $O(\beta)$, to (41) arises from the stationary point (43a). Thus (41) becomes as $\beta \to 0$

$$\eta(X, Z, t) = -C\beta(2D)^{-1} e^{i(MX - \sigma t)} \int_{-\infty}^{mX} \gamma(x', mZ)dx' + O(\beta^2).$$

This is the steady surface shape corresponding to the solution of the initial-value problem (12a–d). The real form of the steady-state surface shape corresponding to the solution of

(2a–c), (5a, b) is, using (4b), (9),

$$-\eta(X, Z, t)/C = \sin(MX - \sigma t) + (\beta/2)\int_{-\infty}^{mX} \gamma(x', mZ)\,dx' \cos(MX - \sigma t) + O(\beta^2). \tag{53}$$

Neglecting $O(\beta^2)$, the formula (53) gives the dependence of the wave phase on the bottom topography. In fact, bringing it into the form

$$-\eta(X, Z, t)/C = a(mX, mZ)\sin[MX - \sigma t + p(mX, mZ)],$$

the function $p$ represents the phase shift in the wave from $-\infty$ to $X$. To first order in $\beta$ it is given by

$$p(mX, mZ) = (\beta/2)\int_{-\infty}^{mX} \gamma(x', mZ)\,dx'.$$

The wave crests will be located on $X - Z$ curves satisfying

$$M(X - X_\infty) + (\beta/2)\int_{-\infty}^{mX} \gamma(x', mZ)\,dx' = 0,$$

where $X_\infty$ is the abscissa of the wave crest for large $|Z|$. For crests that have completely passed over the non-flat part of the bottom, i.e., the support of $\gamma$, one has

$$M(X - X_\infty) + (\beta/2)\int_{-\infty}^{\infty} \gamma(x', mZ)\,dx' = 0, \quad X_\infty \gg 0.$$

Hence the deviation of the crest, for any Z, is proportional to the area under $y = \gamma(x', mZ)$, $-\infty < x' < \infty$. Thus the shape of the wave crest gives an indication of the bottom configuration: they will be retarded on passing over a region of shallow water, $\gamma > 0$, and will advance over a region of deep water, $\gamma < 0$.

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